Another approach to the trisection problem

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The standard way of proving that certain algebraic constructions cannot be obtained with ruler and compass alone is based upon the following theorem (see [1, §4.2]):

Theorem: Let $z_1, z_2, \ldots, z_n$ be complex numbers and let

$$F = \mathbb{Q}(z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n}).$$

Then any complex number z which is constructible from $0, 1, z_1, z_2, \ldots, z_n$ with ruler and compass alone is algebraic over F and its degree over F is a power of 2.

Let A and B be the points corresponding to 0 and 1 respectively on the complex plane. Briefly put, the usual way of using this theorem in order to prove that it is impossible to trisect an arbitary angle with ruler and compass alone, is to observe that, if such a construction were possible, then it would be possible to trisect any angle of the form $\angle BAZ$, where Z is the point of the complex plane corresponding to a number $z \in \mathbb{C}$ such that $|z| = 1$. Such a number z has the form $\cos\theta + i\sin\theta$ for some real number θ. But then the point P of the plane corresponding to $\cos(\theta/3)$ would be constructible from A, B, and Z using ruler and compass alone, since P would be the projection on the real axis of the point that corresponds to $\cos(\theta/3) + i\sin(\theta/3)$. However, since

$$(\forall\theta \in \mathbb{R}) : \cos(\theta) = 4\cos^3(\theta/3) - 3\cos(\theta/3), \tag{1}$$

then, in particular, if we start with $\theta = \pi/3$ we conclude that the polynomial

$$P(x) = 4x^3 - 3x - \tfrac{1}{2}$$

must be reducible over $\mathbb{Q}$, which is not true. This can be proved using two theorems. In order to state the first theorem (see [1, §2.16]), let us introduce the following notation: we will say that a polynomial in one variable with integer coefficients is *primitive* if, and only if, the only common factor of its coefficients is ± 1.

Gauss' lemma: The product of two primitive polynomials is again a primitive polynomial.

As an easy consequence of Gauss' lemma, we have that if $P(x) \in \mathbb{Z}[x]$ has positive degree and is irreducible in $\mathbb{Z}[x]$, then $P(x)$ is also irreducible in $\mathbb{Q}[x]$. So, in order to prove that a polynomial $P(x) \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$, it will be enough to prove that it is irreducible in $\mathbb{Z}[x]$. In order to determine whether or not this is true, it is convenient to use this criterion (see [2, ch. 2]):

Eisenstein's criterion: If $P(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ and there is a prime p such that $p \mid a_i$, $0 \leqslant i < n$, $p \nmid a_n$ and $p^2 \nmid a_0$, then $P(x)$ is irreducible in $\mathbb{Z}[x]$.

In order to use these results to prove that $4x^3-3x-\frac{1}{2}$ is irreducible in $\mathbb{Q}[x]$, observe that the irreducibility of this polynomial is equivalent to the irreducibility of

$$2\cdot\left(4\left(\frac{x}{2}\right)^3-3\frac{x}{2}-\frac{1}{2}\right)=x^3-3x-1.$$

But it follows from Eisentein's criterion (with $p=3$) that the polynomial $P(x+1)(=x^3+3x^2-3)$ is irreducible in $\mathbb{Z}[x]$ and therefore in $\mathbb{Q}[x]$. So, $P(x)$ is also irreducible in $\mathbb{Q}[x]$.

Here we shall see another way of proving the impossibility of trisecting an arbitrary angle with ruler and compass alone, which makes no reference to any trigonometric formula such as (1) (or any equivalent geometric formulation; cf. [3, §I.4]) nor to the polynomial $P(x)$ or any other polynomial equivalent to it (such as x^3-3x-1).

Let us start with the unit circle, i.e., the circle with centre A passing through B (see Figure 1). Consider also the circle with centre B passing through A and let Z be one of the two points at which the two circles meet. Let W be a point of the unit circle such that the angle BAW is one third of the angle BAZ. Since the point Z can be constructed from A and B with ruler and compass alone, it follows from the theorem above that, if the angle BAZ could be trisected with ruler and compass alone, then the complex number w corresponding to the point W would be algebraic over $\mathbb{Q}$ with degree a power of 2. Let us see that this is impossible.

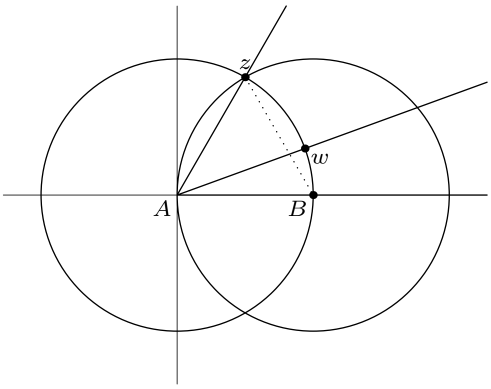

Figure 1

A simple geometrical argument shows that, since the triangle $\triangle ABZ$ is equilateral, then, if z is the complex number that corresponds to Z, $z^3=-1$. But since

$$0=z^3+1=(z+1).(z^2-z+1)$$

and since clearly $z\neq-1$, it follows that $z^2-z+1=0$. On the other hand, it is clear that $w^3=z$, and so w is a zero of the polynomial $P(x)=x^6-x^3+1$. Let us prove that $P(x)$ is irreducible in $\mathbb{Z}[x]$; it will follow from this fact that w is algebraic over $\mathbb{Q}$ with degree 6, which is not a power of 2. One way of proving that $P(x)$ is irreducible in $\mathbb{Z}[x]$ is to apply Eisenstein's criterion to $P(x+a)$ for some well chosen integer a. The fact that $\binom{n}{k}$ is a multiple of 3 whenever n is equal to 3 or 6 and $k\in\{1,2,4,5\}$ shows that the coefficients of x, x^2, x^4, and x^5 in $P(x+a)$ are multiples of 3 and this fact suggests that we might try to choose a such that Eisenstein's criterion applies with $p=3$. The coefficients of x^6, x^3 and the constant term of $P(x+a)$ are equal to 1, $20a^3-1$ and

$a^6 - a^3 + 1$ respectively; therefore, if we take $a = -1$ these values will be -21 and 3 respectively, so Eisenstein's criterion can be applied here with $p = 3$.

Another way of proving that $P(x)$ is irreducible in $\mathbb{Z}[x]$ consists in applying this theorem (cf. [4] for the standard way of stating this theorem and also for the proof):

A. Cohn's generalized theorem: If $Q(x) \in \mathbb{Z}[x]$ has non-negative coefficients and if $Q(x)$ takes a prime value at some $n \in \mathbb{N}$ greater than any coefficient, then $Q(x)$ is irreducible in $\mathbb{Z}[x]$.

Since $2^6 + 2^3 + 1 = 73$, which is a prime number, it follows that $x^6 + x^3 + 1$ is irreducible in $\mathbb{Z}[x]$ and, since the irreducibility of a polynomial $Q(x)$ in $\mathbb{Z}[x]$ is equivalent to the irreducibility of $Q(-x)$, this proves that $P(x)$ is irreducible in $\mathbb{Z}[x]$.

Notice that the measure of the angle BAZ is $\pi/3$. The method described above can also by applied to angles whose measure is $2\pi/3$. In fact, the calculations will be even simpler in that case, since the polynomial whose irreducibility will have to be established will be $x^6 + x^3 + 1$; therefore, A. Cohn's generalized theorem can be used directly.

References

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4. J. Brillhart, M. Filaseta, and A. Odlyzko, On an irreducibility theorem of A. Cohn, *Can. J. Math.*, **33** (1981) pp. 1055–1059.