When is a group homomorphism a covering homomorphism?

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Abstract

Let G be a topological group which acts in a continuous and transitive way on a topological space M. Sufficient conditions are given that assure that, for every $m \in M$, the map from G onto M defined by $g \mapsto g \cdot m$ is an open map. Some consequences of the existence of these conditions, concerning spinor groups and covering homomorphisms between Lie groups, are obtained.

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Introduction

A standard reference concerning Clifford algebras and spinor groups is [2, Part I]. In that article, the authors define, for each $k \in \mathbb{N}$, the spinor group $\mathrm{Spin}(k)$ as a group of invertible elements of the real Clifford algebra C_k. There is a natural continuous homomorphism ρ from $\mathrm{Spin}(k)$ to $SO(k, \mathbb{R})$ and the authors state that ρ is a covering homomorphism (see [2, proposition 3.13]).

However, what is actually proved is just that ρ is surjective and that the kernel has two elements and this is not enough to prove the statement. The same problem arises in [3, §I.6], in [5, §20.2] and in [6, §4.7]. The goal of this article is to state and prove a theorem concerning topological groups that assures that ρ is really a covering homomorphism. Another way of doing this, using Lie theory, can be found in [4, §II.XI]. We also give a new proof of a theorem concerning covering homomorphisms between Lie groups.

1 The main theorem

In what follows, every topological space (and, in particular, every topological group) is Hausdorff. The unit element of a group G will be denoted by e_G or simply by e, when there is only one group involved. The concepts and basic facts concerning topological groups which will be needed here can be found at [4, ch. II].

If φ is a continuous homomorphism from a topological group G to a topological group H, in order that φ is a covering homomorphism it is necessary that φ is surjective and that the kernel of φ is discrete. In general, these conditions are not sufficient to assure that φ is a covering homomorphism. As an example, let α be a real irrational number and let G be the subgroup of the torus $S^1 \times S^1$ whose elements are those of the form $(e^{it}, e^{i\alpha t})$, for some $t \in \mathbb{R}$. Consider the homomorphism of the group $(\mathbb{R}, +)$ onto G that maps each $t \in \mathbb{R}$ into $(e^{it}, e^{i\alpha t})$. If you consider in $\mathbb{R}$ and in G the usual topologies, then this map is a continuous and bijective homomorphism, but it is not a homeomorphism since G is not locally compact. An even simpler example is given by the identity map from $(\mathbb{R}, +)$ (with the discrete topology) onto $(\mathbb{R}, +)$ (with the usual topology).

In order to give general conditions concerning two topological groups G and H that assure that each continuous and surjective homomorphism from G onto H with discrete kernel is a covering homomorphism, we shall have to prove a theorem concerning group actions on topological spaces.

Theorem 1 *Let G be a Lindelöf and locally compact topological group which acts in a continuous and transitive way on a Baire space M. If $m \in M$, then the map*

$$\begin{array}{ccc} G & \longrightarrow & M \\ g & \mapsto & g \cdot m \end{array}$$

is an open map.

Proof: It will be enough to prove that if V is a neighborhood of e, then $V \cdot m$ is a neighborhood of m. Let W be a neighborhood of e such that

$W^{-1}\cdot W \subset V$ and suppose that $W\cdot m$ is a neighborhood of some of its points; in other words, suppose that, for some $w_0 \in W$, $W \cdot m$ is a neighborhood of $w_0 \cdot m$. Then ${w_0}^{-1}\cdot(W\cdot m)$ is a neighborhood of m and therefore $\bigcup_{w\in W} w^{-1}\cdot(W\cdot m)(= V\cdot m)$ is a neighborhood of m.

Therefore, all that remains to be proved is that among all neighborhoods W of e such that $W^{-1}\cdot W \subset V$ there is at least one such that $W\cdot m$ is a neighborhood of some of its points, and this is equivalent to saying that the interior of $W\cdot m$ is not empty. Let W be a compact neighborhood of e such that $W^{-1}\cdot W \subset V$; such a neighborhood exists since we are supposing that G is locally compact. It is clear that the interior of $W\cdot m$ is not empty if and only if, for some $g \in G$, the interior of $g\cdot(W\cdot m)$ is not empty. It follows from the fact that G is a Lindelöf space and from the fact that $\bigcup_{g\in G} g\cdot \mathring{W} = G$ that there is a sequence $(g_n)_{n\in\mathbb{N}}$ of elements of G such that $\bigcup_{n\in\mathbb{N}} g_n\cdot W = G$ and, therefore, such that $\bigcup_{n\in\mathbb{N}} g_n\cdot(W\cdot m) = M$, since the action of G on M is transitive. For each $n\in\mathbb{N}$, $g_n\cdot(W\cdot m)$ is a compact set, since W is compact and the action is continuous, and, in particular, each set $g_n\cdot(W\cdot m)$ is a closed set. Since M is a Baire space, there is at least one $n\in\mathbb{N}$ such that the interior of $g_n\cdot(W\cdot m)$ is not empty and, as it has already been observed, this is equivalent to the assertion that the interior of $W\cdot m$ is not empty. □

This proof is adapted from the proof of the corollary in [1, §9] (see the corollary 2 below).

It should be observed that if G is a connected and locally compact topological group, then G is also a Lindelöf space. In fact, since G is connected, it is generated by any neighborhood of e (see [4, §II.IV, theorem 1]) and therefore if V is a compact neighborhood of e then $G = \bigcup_{n\in\mathbb{N}} V^n$. This proves that G is σ-compact and therefore Lindelöf. Of course, it follows from this observation and from the fact that any connected component of a topological group is homeomorphic to the connected component of the unit element that, more generally, if a locally compact group G has only a finite or countable set of connected components, then G is Lindelöf.

Before we proceed, let us see an interesting consequence of the previous theorem. This corollary is the corollary of [1, §9] that was mentioned above; we prove it for completeness and because the proof is very short.

Corollary 2 *Let G be a Lindelöf and locally compact group which acts in a continuous and transitive way on a Baire space M. If $m \in M$, if H is the stabilizer of m in G and if in G/H one considers the final topology with respect to the natural projection from G onto G/H, then the map*

$$\begin{array}{ccc} G/H & \longrightarrow & M \\ gH & \mapsto & g\cdot m \end{array}$$

is a homeomorphism.

Proof: The map is clearly a continuous bijection and all that remains to be proved is that it is an open map. If A is an open set of G/H and $\pi : G \longrightarrow G/H$ denotes the natural projection, then A is mapped onto $\pi^{-1}(A) \cdot m$ and this set is an open set, by the previous theorem. $\square$

Theorem 3 *Let G and H be topological groups and suppose that, as topological spaces, G is Lindelöf and locally compact and H is a Baire space. If φ is a continuous homomorphism from G onto H, then φ is a covering homomorphism if and only if its kernel is discrete.*

Proof: The homomorphism φ induces the action from G on H defined by

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}(H) \\ g & \mapsto & \left(\begin{array}{ccc} H & \longrightarrow & H \\ h & \mapsto & \varphi(g) \cdot h \end{array}\right). \end{array}$$

This action is continuous (since φ is continuous) and transitive (since φ is surjective). Therefore, it follows from the theorem 1 (with $m = e_H$) that φ is an open map. Let V be a neighborhood of e_G such that $V \cap \ker\varphi = \{e_G\}$, let W be an open neighborhood of e_G such that $W \cdot W^{-1} \subset V$ and define $W' = \varphi(W)$. Since φ is an open map, W' is a neighborhood of e_H. Then

$$\varphi^{-1}(W') = \bigcup_{g \in \ker\varphi} g \cdot W$$

and, furthermore, this is a disjoint union, because if $g, h \in \ker\varphi$ and $v, w \in W$ are such that $g \cdot v = h \cdot w$, then $v \cdot w^{-1} = g^{-1} \cdot h \in \ker\varphi$; since $v \cdot w^{-1} \in V$, it follows that $g = h$. Therefore $\varphi^{-1}(W')$ is homeomorphic to $\ker(\varphi) \times W'$ when we consider in $\ker\varphi$ the discrete topology. This proves that φ is a covering homomorphism. $\square$

In order to apply this theorem to the spinor groups, it will be enough to prove that these groups are Lindelöf and locally compact. But it is a consequence of the definition of $\mathrm{Spin}(k)$ (see [2, pp. 6–8]) that this group can be seen as a closed subset of a finite-dimensional real vector space (with the usual topology); therefore, it is both a Lindelöf space and a locally compact space. Since $SO(k, \mathbb{R})$ is compact (and therefore a Baire space) the natural homomorphism from $\mathrm{Spin}(k)$ onto $SO(k, \mathbb{R})$ is a covering homomorphism. As it was observed before (see [2, Part I] and [3, §I.6]), this fact can be used to prove that $\mathrm{Spin}(k)$ has a Lie group structure.

2 Lie group homomorphisms

Let us extract another consequence of theorem 3. If φ is an analytic homomorphism from a Lie group G to a Lie group H, let φ^* denote the differential of φ at e_G. Note that, since every connected Lie group is locally compact, Lindelöf and a Baire space, theorem 3 implies that an analytic homomorphism φ from a connected Lie group G to a Lie group H is a covering homomorphism if and only if φ is surjective and $\ker\varphi$ is discrete.

Theorem 4 *If G and H are connected Lie groups and φ is an analytic homomorphism from G onto H, then φ is a covering homomorphism if and only if φ^* is an isomorphism.*

Proof: Using the exponential map it is easy to prove that if φ^* is surjective then φ is also surjective. In fact, these statements are equivalent. If φ is surjective, then it induces a bijective analytic homomorphism $\psi : G/\ker(\varphi) \longrightarrow H$. It is in fact a homeomorphism; this can be seen as a consequence of corollary 2 or as an application of the theorem of invariance of domain. Since every continuous homomorphism between Lie groups is analytic (see [4, §IV.XIII] or [7, theorem 3.39]), it follows that ψ^{-1} is also analytic. Therefore, ψ^* is an isomorphism and this implies that φ^* is surjective; in fact, if π denotes the natural projection from G onto $G/\ker(\varphi)$, then π^* is surjective and

$$\varphi = \psi \circ \pi \Longrightarrow \varphi^* = \psi^* \circ \pi^*.$$

Finally, observe that φ^* is injective if and only if the kernel of φ is discrete. Indeed, if φ^* is not injective, then there is some X in the Lie algebra $\mathfrak{g}$ of G such that $X \neq 0$ and that $\varphi^*(X) = 0$, and this would imply that

$$(\forall t \in \mathbb{R}) : \varphi(\exp(tX)) = \exp(t\varphi^*(X)) = e_H.$$

On the other hand, if φ^* is injective and if U is neighborhood of 0 in $\mathfrak{g}$ such that $\exp|_U$ and $\exp|_{\varphi^*(U)}$ are injective and that $\exp(U)$ is a neighborhood V of e_G, then every $g \in V \setminus \{e_G\}$ has the form $\exp(X)$ for some $X \in U \setminus \{0\}$ and therefore

$$\varphi(g) = \varphi(\exp(X)) = \exp(\varphi^*(X));$$

since $\varphi^*(X) \neq 0$ and $\exp|_{\varphi^*(U)}$ is injective, this proves that $\varphi(g) \neq e_H$. □

Cf. [7, p. 100] for another proof of this theorem.

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